a review of Decomposition spaces, incidence algebras and Mobius inversion. I: Basic theory by Galvez-Carrillo, Imma; Kock, Joachim; Tonks, Andrew

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Decomposition spaces, incidence algebras and Möbius inversion. I: Basic theory. (English)

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The authors wrote out a long manuscript [*I. Gálvez-Carrillo* et al., "Decomposition spaces, incidence algebras and Möbius inversion", Preprint, arXiv:1404.3202], which has been divided reasonably into six papers of more manageable size. The first two sections of that long manuscript constitute the present paper, which is the first part of a trilogy followed by [*I. Gálvez-Carrillo* et al., Adv. Math. 333, 1242–1292 (2018; Zbl 1403.18016); ibid. 334, 544–584 (2018; Zbl 1403.18017)]. The long appendix of that long manuscript has become an independent paper [*I. Gálvez-Carrillo* et al., "Homotopy linear algebra", Preprint, arXiv:1602.05082] devoted to homotopy linear algebra. The offshoot [*I. Gálvez-Carrillo* et al., "Decomposition spaces and restriction species", Preprint, arXiv:1708.02570] shows that Schmitt coalgebras of restriction species naturally define decomposition spaces, introducing a new notion of *directed restriction species*, of which the Butcher-Connes-Kreimer bialgebra and the Manchon-Manin bialgebra of directed graph are examples. The excrescence [*I. Gálvez-Carrillo* et al., "Decomposition spaces in combinatorics", Preprint, arXiv:1612.09225] gives such examples as the binomial posets in [*P. Doubilet* et al., in: Proc. 6th Berkeley Sympos. math. Statist. Probab., Univ. Calif. 1970, 2, 267–318 (1972; Zbl 0267.05002)], the Faà di Bruno bialgebra, the Butcher-Connes-Kremier bialgebra of trees and Hall algebras.

This paper is the first part of a trilogy devoted to the theory of *decomposition spaces*, which are simplicial ∞- groupoids abiding by a certain exactness condition. The principal objective in this paper is to introduce decomposition spaces as a general framework for incidence algebra and Möbius inversion. The second part [*I. Gálvez-Carrillo* et al., Adv. Math. 333, 1242–1292 (2018; Zbl 1403.18016)] arrives at the notion of Möbius decomposition space as a far-reaching generalization of the notion of Möbius category in [*P. Leroux*, Cah. Topologie Géom. Différ. Catégoriques 16, 280–282 (1976; Zbl 0364.18001)]. The third part introduces the Möbius decomposition space of Möbius intervals, subsuming discoveries by *F. W. Lawvere* and *M. Menni* [Theory Appl. Categ. 24, 221–265 (2010; Zbl 1236.18001)].

The authors generalize the familiar notion of incidence algebra of a locally finite poset in three directions:

(1) replacing posets by categories and ∞-categories;

(2) replacing scalar coefficients by ∞-groupoids;

(3) replacing the Segal condition by a weaker one that still allows the construction of incidence algebra.

They then arrive at decomposition spaces as a systematic framework for decomposing structures, while categories are the systematic framework for composing structures. They prefer incidence *coalgebras* to incidence algebras, which are merely the convolution algebras determined by their linear duals. The principal discovery is a weaker condition (called the *decomposition axiom*) than the Segal or Rezk condition allowing the construction of a coassociative incidence coalgebra and a Möbius inversion principle. An important advantage of having the classical settings of posets and monoids on the same footing is that they may then be connected by an appropriate class of functors. They are called the CULF (C for "conservative" and ULF for "unique lifting of cofactorizations") functors between decomposition spaces, and induce coalgebra homomorphisms. Decomposition spaces were discovered first by *T. Dyckerhoff* and *M. Kapranov* ["Higher Segal spaces. I", Preprint, arXiv:1212.3563], who called them unital 2-Segal spaces. The authors, unaware of their work, have arrived at the same notion and have developed the theory that are mostly orthogonal to theirs. The definitions are different in guise: the definition of decomposition space refers to preservation of certain pullbacks, while the definition of 2-Segal space is concerned with triangulations of convex polygons. The authors were inspired by rather elementary aspects of combinatorics and quantum field theory, whereas Dyckerhoff and Kapranov were motivated by representation theory, goemetry and homological algebra. The authors' examples are drawn from incidence algebra and Möbius inversion, while Dyckerhoff and Kapranov have in mind cyclic bar construction, mapping class groups surface geometry besides Hall algebras and Hecke ones.

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MSC:

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